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On properties of generalized quadratic operators☆

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In this paper, we investigate the set $\omega(P)$ of generalized quadratic operators A satisfying the equation $A^2 = \alpha A + \beta P$ for all complex numbers α and β and for an idempotent operator P such that $AP = PA = A$. Furthermore, the close relationship between the operator $A \in \omega(P)$ and the idempotent operator P are established and expressions for the inverse, the Moore–Penrose inverse and the Drazin inverse of $A \in \omega(P)$ are given. Some related results are also obtained.

1. Introduction

Generalized quadratic operators and idempotents are becoming important tools in several areas of mathematics, statistics and engineering. This phenomenon is illustrated in many papers on integral equations, iterative methods in numerical linear algebra, signal processing and linear regression [1,2,6–14]. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$. An idempotent P is called an orthogonal projection if $P^2 = P = P^*$, where P^* is the adjoint of P. For an operator $T \in \mathcal{B}(\mathcal{H})$ and a closed subspace $\mathcal{K}$ of $\mathcal{H}$, $\mathcal{R}(T)$, $\mathcal{N}(T)$, $\sigma(T)$, $acc[\sigma(T)]$, $T|_{\mathcal{K}}$ and $I_{\mathcal{K}}$ denote the range, the null space, the spectrum of T, the

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accumulation points of $\sigma(T)$, the restriction of T on $\mathcal{K}$ and the identity on $\mathcal{K}$ (or simply, I if there does not exist confusion), respectively. The set $\omega(P)$ is defined as

$$\omega(P) = \{A \in \mathcal{B}(\mathcal{H}) : A^2 = \alpha A + \beta P, AP = PA = A, P^2 = P, \forall \alpha, \beta \in \mathbb{C}\}.$$

Following [8], we call $A \in \omega(P)$ generalized quadratic operator with respect to P. In [1], a subclass of $\omega(I_n)$ was investigated by Aleksiejczyk and Smoktunowicz, where A satisfies the quadratic equation $(A - \alpha I_n)(A - \beta I_n) = 0$ with $\alpha, \beta \in \mathbb{C}$. Later on, Farebrother and Trenkler [8] extended the concept considered by Radjavi and Rosenthal [11]. They investigated the set $\omega(P)$ of square matrices $A \in \mathbb{C}^{n\times n}$ and paid special attention to the Moore–Penrose and group inverse of matrices belonging to $\omega(P)$. Many generalizations and applications came later. The reader refers to the papers by Benítez and Thome [2], Du et al. [5], Choi and Wu [6], Fang et al. [7], Özdemir and Özban [9], Zhou and Wang [12], which have a complete panorama on these matters.

This paper is devoted to the natural generalization of $\omega(P)$ to the infinite dimensional setting. Some new relations between the operator $A \in \omega(P)$ and the idempotent operator P are obtained. By using the technique of block operator matrices, we give explicit expressions for the Moore–Penrose inverse A^+, the Drazin inverse A^D and $(A - B)^{-1}$in terms of A, B, P and Q for $A \in \omega(P)$ and $B \in \omega(Q)$. Meanwhile, the spectral characterizations of generalized quadratic operators are obtained. These results will lead us to understand the closed relationship between the operator $A \in \omega(P)$ and the idempotent operator P.

2. Preliminaries and auxiliary results

In what follows $T^{\frac{1}{2}}$ denotes the positive square root of T if T is positive. First, we state some useful results.

Lemma 2.1. *Let $P, Q \in \mathcal{B}(\mathcal{H})$ with $P = P^2$ and $Q = Q^2 = Q^*$.*

(1) *If $P - Q$ is invertible, then $\mathcal{N}(I_{\mathcal{R}(P)} - Q|_{\mathcal{R}(P)}) = \{0\}, \mathcal{N}(Q|_{\mathcal{R}(P)^\perp}) = \{0\}$.*

(2) [3, Lemma 1.1] *There exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $S^{-1}PS$ is an orthogonal projection.*

(3) [10, Theorem 2.1] *If $\mathcal{R}(P) = \mathcal{R}(Q)$, then $P + P^* - I$ is always invertible and*

$$Q = P(P + P^* - I)^{-1} = (P + P^* - I)^{-1}P^*.$$

Lemma 2.2. *Let $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$ and $\beta \neq 0$, $B \in \omega(Q)$ such that $B^2 = mB + nQ$ and $n \neq 0$.*

(1) *$\mathcal{R}(A) = \mathcal{R}(P)$ is closed, $\mathcal{N}(A) = \mathcal{N}(P)$.*

(2) *$\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(PQ)$ is closed.*

Proof. (1) If $\beta \neq 0$, from $A^2 = \alpha A + \beta P$ it follows $\beta P = A(A - \alpha I)$.This implies $\mathcal{R}(P) \subset \mathcal{R}(A)$. Further, since P is idempotent and $A = PA$, $\mathcal{R}(P)$ is closed and $\mathcal{R}(A) \subset \mathcal{R}(P)$. Thus $\mathcal{R}(A) = \mathcal{R}(P)$ is closed. From $AP = PA = A$, A and P can be written as

$$A = \begin{pmatrix} A_1 & A_1P_1 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \tag{1}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp$. From $A^2 = \alpha A + \beta P$, we get that $A_1^2 = \alpha A_1 + \beta I$. Therefore A_1 is invertible with

$$A_1^{-1} = \frac{1}{\beta}(A_1 - \alpha I).$$

If there exists a vector $x = x_1 + x_2$ with $x_1 \in \mathcal{R}(P)$ and $x_2 \in \mathcal{R}(P)^\perp$ such that $x \in \mathcal{N}(A)$, then $x_1 + P_1x_2 = 0$, i.e., $x \in \mathcal{N}(P)$. On the other hand, $A = AP$ implies that $\mathcal{N}(P) \subset \mathcal{N}(A)$. Hence $\mathcal{N}(P) = \mathcal{N}(A)$.

(2) Note that $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*)$ is closed. If $\mathcal{R}(A^*)$ is closed, then $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp = \mathcal{N}(P)^\perp = \mathcal{R}(P^*)$. So

$$\begin{aligned}\mathcal{R}(AB) \text{ is closed} &\Longleftrightarrow \mathcal{R}(AP_{\mathcal{R}(B)}) \text{ is closed} \Longleftrightarrow \mathcal{R}(P_{\mathcal{R}(B)}A^*) \text{ is closed}\\ &\Longleftrightarrow \mathcal{R}(P_{\mathcal{R}(B)}P_{\mathcal{R}(A^*)}) \text{ is closed} \Longleftrightarrow \mathcal{R}(P_{\mathcal{R}(Q)}P_{\mathcal{R}(P^*)}) \text{ is closed}\\ &\Longleftrightarrow \mathcal{R}((Q+Q^*-I)^{-1}Q^*P^*(P+P^*-I)^{-1}) \text{ is closed}\\ &\Longleftrightarrow \mathcal{R}(Q^*P^*) \text{ is closed} \Longleftrightarrow \mathcal{R}(PQ) \text{ is closed.} \quad \square\end{aligned}$$

When consider the matrix representation of $A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$, we need the following results.

Lemma 2.3. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a bounded linear operator on* $\mathcal{H} \oplus \mathcal{K}$.

(1) [13, Theorem 2.1] *If* A_{11} *is invertible, then* A *is invertible if and only if the Schur complement* $A_{22} - A_{21}A_{11}^{-1}A_{12}$ *is invertible*.

(2) [5, Theorem 1] *If* A *is orthogonal projection with* A_{11} *and* A_{22} *injective, then there exists a unitary operator* D *from* $\mathcal{K}$ *into* $\mathcal{H}$ *such that* $A_{12} = A_{21}^* = A_{11}^{\frac{1}{2}}(I - A_{11})^{\frac{1}{2}}D, A_{22} = D^*(I - A_{11})D$.

We also need the following well-known criteria about range.

Lemma 2.4 [4]. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be a positive operator. Then* $\mathcal{R}(A) \subseteq \mathcal{R}\left(A^{\frac{1}{2}}\right)$. $\mathcal{R}(A)$ *is closed if and only if* $\mathcal{R}(A) = \mathcal{R}\left(A^{\frac{1}{2}}\right)$. $\mathcal{R}(A) = \mathcal{H}$ *if and only if* A *is invertible*.

3. Generalized quadratic operators

In this section we characterize generalized quadratic operators on arbitrary Hilbert spaces. Let $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$. Then $\omega(P)$ contains the following subclasses.

(1) A is 2-nilpotent if $\alpha = 0$ and $\beta = 0$; $\frac{A}{\alpha}$ is idempotent if $\alpha \neq 0$ and $\beta = 0$; $A^2 = \beta P$ if $\alpha = 0$ and $\beta \neq 0$.

(2) $A = \frac{\alpha}{2}P + N$ with $NP = PN = N$ and $N^2 = 0$ if $\beta \neq 0$ and $\alpha^2 + 4\beta = 0$.

Furthermore, if $\beta \neq 0$ and $\alpha^2 + 4\beta \neq 0$. we show that there exists an invertible operator S such that $S^{-1}AS$ and $S^{-1}PS$ are diagonal operators.

Theorem 3.1. *Let* $\beta \neq 0$ *and* $\alpha^2 + 4\beta \neq 0$. *Then* $A \in \omega(P)$ *such that* $A^2 = \alpha A + \beta P$ *if and only if*

(1) $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } \lambda^2 = \alpha\lambda + \beta\}$;

(2) *There exists a resolution* $\{E(\lambda) : \lambda \in \sigma(A)\}$ *of the identity* I *and an invertible operator* S *such that*

$$SAS^{-1} = \sum_{\lambda\in\sigma(A)} \oplus\lambda E(\lambda), \quad SPS^{-1} = \sum_{\lambda\in\sigma(A)} \oplus E(\lambda),$$

where $\oplus$ *denotes the orthogonal direct sum*. $E(\lambda), \lambda \in \sigma(A)$ *are orthogonal projections adding up to unity*, $\sum_{\lambda\in\sigma(A)} E(\lambda) = I$. *And* $E(\lambda)E(\mu) = E(\mu)E(\lambda) = 0$ *if* $\lambda, \mu \in \sigma(A)$ *and* $\lambda \neq \mu$.

Proof. (1) Since $A^2 = \alpha A + \beta P$ and $AP = PA = A$, $A^3 = \alpha A^2 + \beta A$. Let $\lambda \in \sigma(A)$. Then $A^3 = \alpha A^2 + \beta A$ implies that $\lambda^3 = \alpha\lambda^2 + \beta\lambda$ by the spectral mapping theorem. Hence, $\lambda = 0$ or $\lambda^2 = \alpha\lambda + \beta$, i.e.,

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } \lambda^2 = \alpha\lambda + \beta\}.$$

(2) If $\beta \neq 0$, by the proof of Lemma 2.2, Eq. (1), we know there exists an invertible operator $S_1 = \begin{pmatrix} I & P_1 \\ 0 & I \end{pmatrix}$ such that

$$S_1AS_1^{-1} = A_1 \oplus 0, \quad S_1PS_1^{-1} = I \oplus 0, \quad \sigma(A_1) \subseteq \{\lambda : \lambda^2 = \alpha\lambda + \beta\}. \tag{2}$$

Since $\alpha^2 + 4\beta \neq 0$, there exist complex numbers $\lambda_1 \neq \lambda_2$such that $\sigma(A_1) \subseteq \{\lambda_1, \lambda_2\}$. For each $\lambda_i \in \sigma(A_1)$, $i = 1, 2$, define the Riesz projection of A_1 associated to λ_i by

$$F(\lambda_i) = \frac{1}{2\pi i}\int_{C_{\lambda_i}} (\mu I - A_1)^{-1}\, d\mu,$$

where C_{λ_i} is a smooth closed curve such that λ_i is contained in the interior of C_{λ_i} and $\sigma(A_1)\setminus\{\lambda_i\}$ is contained in the exterior of C_{λ_i}. Since $\lambda_i \in \sigma(A_1)$ is a simple pole of the resolution $(\mu I - A_1)^{-1}$, $A_1 = \lambda_1 F(\lambda_1) \dot{+} \lambda_2 F(\lambda_2)$, where $\dot{+}$ denotes the algebraical direct sum, $F(\lambda_i)^2 = F(\lambda_i)$ for $i = 1, 2$, $F(\lambda_1)\dot{+}F(\lambda_2) = I_{\mathcal{R}(P)}$ and $F(\lambda_1)F(\lambda_2) = F(\lambda_2)F(\lambda_1) = 0$.

Since $F(\lambda_i)\mathcal{R}(P)$ is an invariant subspace for A_1, operator A_1 can be written as $A_1 = \begin{pmatrix} \lambda_1 I & A_{12} \\ 0 & \lambda_2 I \end{pmatrix}$ with the space decomposition $\mathcal{R}(P) = F(\lambda_1)\mathcal{R}(P) \oplus (\mathcal{R}(P) \ominus F(\lambda_1)\mathcal{R}(P))$. Put $\widetilde{X} = \begin{pmatrix} I & \frac{A_{12}}{\lambda_1-\lambda_2} \\ 0 & I \end{pmatrix}$. Then

$$\begin{pmatrix} I & \frac{A_{12}}{\lambda_1-\lambda_2} \\ 0 & I \end{pmatrix}\begin{pmatrix} \lambda_1 I & A_{12} \\ 0 & \lambda_2 I \end{pmatrix}\begin{pmatrix} I & -\frac{A_{12}}{\lambda_1-\lambda_2} \\ 0 & I \end{pmatrix} = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix}.$$

Put $S = S_2 S_1$ with $S_2 = \widetilde{X} \oplus I$. Then

$$SAS^{-1} = \sum_{\lambda\in\sigma(A)} \oplus \lambda E(\lambda), \quad SPS^{-1} = \sum_{\lambda\in\sigma(A)} \oplus E(\lambda),$$

where $E(\lambda) = I_{F(\lambda)\mathcal{R}(P)}$ if $\lambda \in \sigma(A_1)$, and $E(0) = I_{\mathcal{R}(P)^\perp}$ if $0 \in \sigma(A)$, are orthogonal projections adding up to unity, $\sum_{\lambda\in\sigma(A)} E(\lambda) = I$. And $E(\lambda)E(\mu) = E(\mu)E(\lambda) = 0$ if $\lambda, \mu \in \sigma(A)$ and $\lambda \neq \mu$.

Conversely, suppose that the statements (1) and (2) hold, we have $\lambda^2 = \alpha\lambda + \beta$ for every $\lambda \in \sigma(A)\setminus\{0\}$. If there exists an invertible operator S such that

$$SAS^{-1} = \sum_{\lambda\in\sigma(A)} \oplus \lambda E(\lambda), \quad SPS^{-1} = \sum_{\lambda\in\sigma(A)} \oplus E(\lambda),$$

then $P^2 = P, AP = PA = A$ and

$$\begin{aligned} SA^2S^{-1} = {} & (SAS^{-1})^2 = 0I_{\mathcal{R}(P)^\perp} \oplus \sum_{\lambda\in\sigma(A)\setminus\{0\}} \oplus \lambda^2 E(\lambda) \\ & = 0I_{\mathcal{R}(P)^\perp} \oplus \sum_{\lambda\in\sigma(A)\setminus\{0\}} \oplus (\alpha\lambda+\beta) E(\lambda) \\ & = 0I_{\mathcal{R}(P)^\perp} \oplus \alpha \sum_{\lambda\in\sigma(A)\setminus\{0\}} \oplus \lambda E(\lambda) + \beta \sum_{\lambda\in\sigma(A)\setminus\{0\}} \oplus E(\lambda) \\ & = \alpha SAS^{-1} + \beta SPS^{-1}. \end{aligned}$$

Hence $A^2 = \alpha A + \beta P$. $\square$

Let $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$ and $\beta \neq 0$, $B \in \omega(Q)$ such that $B^2 = mB + nQ$ and $n \neq 0$. As we have seen, the properties of A and B are closely related with the properties of P and Q. In fact, we can prove that the invertibility of $A - B$ is completely defined by the invertibility of $P - Q$.

Theorem 3.2. *Let P and Q in $\mathcal{B}(\mathcal{H})$ be two idempotents, $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$ and $\beta \neq 0$, $B \in \omega(Q)$ such that $B^2 = mB + nQ$ and $n \neq 0$. If $P - Q$ is invertible, then $A - B$ is invertible.*

Proof. Let $A \in \omega(P)$ and $B \in \omega(Q)$. If $Q = 0$ (or $P = 0$), then $B = 0$(or $A = 0$) since $B = BQ = QB$. The result holds by the proof of Lemma 2.2, item (1).

Let $P \neq 0$ and $Q \neq 0$. Since $S^{-1}AS \in \omega(S^{-1}PS)$ and $S^{-1}BS \in \omega(S^{-1}QS)$, by Lemma 2.1, item 2, to consider the invertibility of $A - B$, without loss of generality, we can assume that Q is an orthogonal projection. Let

$$\mathcal{H}_1 = \mathcal{N}(Q|_{\mathcal{R}(P)}), \quad \mathcal{H}_3 = \mathcal{R}(P)^{\perp} \ominus \mathcal{N}(I_{\mathcal{R}(P)^{\perp}} - Q|_{\mathcal{R}(P)^{\perp}})$$

$$\mathcal{H}_2 = \mathcal{R}(P) \ominus \mathcal{N}(Q|_{\mathcal{R}(P)}), \quad \mathcal{H}_4 = \mathcal{N}(I_{\mathcal{R}(P)^{\perp}} - Q|_{\mathcal{R}(P)^{\perp}}).$$

By Lemma 2.3, item (2), if $P - Q$ is invertible, P and Q can be written in the forms of

$$P = \begin{pmatrix} I & & P_{13} & P_{14} \\ & I & P_{23} & P_{24} \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{and}$$

$$Q = \begin{pmatrix} 0 & & & \\ & Q_{11} & Q_{11}^{\frac{1}{2}}(I - Q_{11})^{\frac{1}{2}}D & \\ & D^*Q_{11}^{\frac{1}{2}}(I - Q_{11})^{\frac{1}{2}} & D^*(I - Q_{11})D & \\ & & & I \end{pmatrix} \tag{3}$$

with respect to the space decomposition $\mathcal{H} = \oplus_{i=1}^{4}\mathcal{H}_i$, where Q_{11} and $I - Q_{11}$ are injective positive contraction operators, D is a unitary operator from $\mathcal{H}_3$ onto $\mathcal{H}_2$ and the entries omitted in (3) are zero. If $P - Q$ is invertible, then

$$\mathcal{H}_3 = \mathcal{R}(D^*Q_{11}^{\frac{1}{2}}(I - Q_{11})^{\frac{1}{2}}, D^*(I - Q_{11})D) \subseteq \mathcal{R}(D^*(I - Q_{11})^{\frac{1}{2}}D) \subseteq \mathcal{H}_3.$$

By Lemma 2.5, $D^*(I - Q_{11})D$ is invertible. Put

$$M_0 = Q_{11}^{\frac{1}{2}}(I - Q_{11})^{\frac{1}{2}}D, \quad M = Q_{11}^{\frac{1}{2}}(I - Q_{11})^{-\frac{1}{2}}D, \quad N = D^*(I - Q_{11})D.$$

By Lemma 2.3, item (1), if $P - Q$ is invertible, then the Schur complement

$$\begin{aligned} & I - Q_{11} - [P_{23} - Q_{11}^{\frac{1}{2}}(I - Q_{11})^{\frac{1}{2}}D]D^*(I - Q_{11})^{-1}DD^*Q_{11}^{\frac{1}{2}}(I - Q_{11})^{\frac{1}{2}} \\ &= I - P_{23}D^*(I - Q_{11})^{-\frac{1}{2}}Q_{11}^{\frac{1}{2}} \\ &= I - P_{23}M^* \quad \text{is invertible.} \end{aligned}$$

Since P has the form (3), by (1), A can be represented as

$$A = \begin{pmatrix} A_1 & A_1P_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{11}P_{13} + A_{12}P_{23} & A_{11}P_{14} + A_{12}P_{24} \\ A_{21} & A_{22} & A_{21}P_{13} + A_{22}P_{23} & A_{21}P_{14} + A_{22}P_{24} \\ & & 0 & \\ & & & 0 \end{pmatrix}, \tag{4}$$

where $A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is invertible and $P_1 = \begin{pmatrix} P_{13} & P_{14} \\ P_{23} & P_{24} \end{pmatrix}$. Let $B = (B_{ij})_{1 \leqslant i,j \leqslant 4}$. From $B = BQ = QB$, we obtain

$$\begin{aligned} B &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{22}Q_{11} + B_{23}M_0^* & B_{22}M_0 + B_{23}N & B_{24} \\ 0 & B_{32}Q_{11} + B_{33}M_0^* & B_{32}M_0 + B_{33}N & B_{34} \\ 0 & B_{42}Q_{11} + B_{43}M_0^* & B_{42}M_0 + B_{43}N & B_{44} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{11}B_{22} + M_0B_{32} & Q_{11}B_{23} + M_0B_{33} & Q_{11}B_{24} + M_0B_{34} \\ 0 & M_0^*B_{22} + NB_{32} & M_0^*B_{23} + NB_{33} & M_0^*B_{24} + NB_{34} \\ 0 & B_{42} & B_{43} & B_{44} \end{pmatrix}. \end{aligned}$$

Since $I-Q_{11}$ is invertible, comparing the two sides of the above equations, we have

$$\begin{cases} B_{22}=B_{22}Q_{11}+B_{23}M_0^*,\\ B_{23}=Q_{11}B_{23}+M_0B_{33},\\ B_{24}=Q_{11}B_{24}+M_0B_{34},\\ B_{32}=B_{32}Q_{11}+B_{33}M_0^*,\\ B_{42}=B_{42}Q_{11}+B_{43}M_0^*. \end{cases} \Longrightarrow \begin{cases} B_{23}=Q_{11}^{\frac{1}{2}}(I-Q_{11})^{-\frac{1}{2}}DB_{33}=MB_{33},\\ B_{22}=B_{23}D^*Q_{11}^{\frac{1}{2}}(I-Q_{11})^{-\frac{1}{2}}=MB_{33}M^*,\\ B_{24}=Q_{11}^{\frac{1}{2}}(I-Q_{11})^{-\frac{1}{2}}DB_{34}=MB_{34},\\ B_{32}=B_{33}D^*Q_{11}^{\frac{1}{2}}(I-Q_{11})^{-\frac{1}{2}}=B_{33}M^*,\\ B_{42}=B_{43}D^*Q_{11}^{\frac{1}{2}}(I-Q_{11})^{-\frac{1}{2}}=B_{43}M^*. \end{cases}$$

Hence

$$B=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & MB_{33}M^* & MB_{33} & MB_{34}\\ 0 & B_{33}M^* & B_{33} & B_{34}\\ 0 & B_{43}M^* & B_{43} & B_{44} \end{pmatrix}. \tag{5}$$

Put $T=I\oplus\begin{pmatrix} I & M\\ 0 & I\end{pmatrix}\oplus I$. Then

$$Q=T\left[0\oplus 0\oplus D^*(I-Q_{11})D\oplus I\right]T^*,\quad B=T\left[0\oplus 0\oplus\begin{pmatrix} B_{33} & B_{34}\\ B_{43} & B_{44}\end{pmatrix}\right]T^*.$$

Since $B^2=mB+nQ$ and T is invertible, we get

$$\begin{aligned} & m\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}+n\begin{pmatrix} 0 &&&\\ & 0 &&\\ && D^*(I-Q_{11})D &\\ &&& I\end{pmatrix}\\ &=\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}T^*T\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}\\ &=\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}\begin{pmatrix} 0 &&&\\ & I & M &\\ & M^* & D^*(I-Q_{11})^{-1}D &\\ &&& I\end{pmatrix}\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}\\ &=\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}\begin{pmatrix} 0 &&&\\ & 0 &&\\ && D^*(I-Q_{11})^{-1}D &\\ &&& I\end{pmatrix}\begin{pmatrix} 0 &&&\\ & 0 &&\\ && B_{33} & B_{34}\\ && B_{43} & B_{44}\end{pmatrix}. \end{aligned}$$

Denote by $B_1=\begin{pmatrix} B_{33} & B_{34}\\ B_{43} & B_{44}\end{pmatrix}$, $Q_1=\begin{pmatrix} D^*(I-Q_{11})D & \\ & I\end{pmatrix}$. Then $B_1Q_1^{-1}B_1=mB_1+nQ_1$. Since $n\neq 0$, B_1 is invertible and

$$B_1^{-1}=\frac{1}{n}(Q_1^{-1}B_1Q_1^{-1}-mQ_1^{-1}). \tag{6}$$

By (4) and (5),

$$A-B=\begin{pmatrix} A_{11} & A_{12} & A_{11}P_{13}+A_{12}P_{23} & A_{11}P_{14}+A_{12}P_{24}\\ A_{21} & A_{22}-MB_{33}M^* & A_{21}P_{13}+A_{22}P_{23}-MB_{33} & A_{21}P_{14}+A_{22}P_{24}-MB_{34}\\ 0 & -B_{33}M^* & -B_{33} & -B_{34}\\ 0 & -B_{43}M^* & -B_{43} & -B_{44} \end{pmatrix}. \tag{7}$$

Define invertible operators G and S by

$$G = \begin{pmatrix} I & & & \\ & I & -M & \\ & & I & \\ & & & I \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I & P_{13}M^*(I - P_{23}M^*)^{-1} & & \\ & (I - P_{23}M^*)^{-1} & & \\ & -M^*(I - P_{23}M^*)^{-1} & -I & \\ & & & -I \end{pmatrix}. \tag{8}$$

Then

$$\begin{aligned} G(A - B)S &= \begin{pmatrix} A_{11} & A_{12} & -(A_{11}P_{13} + A_{12}P_{23}) & -(A_{11}P_{14} + A_{12}P_{24}) \\ A_{21} & A_{22} & -(A_{21}P_{13} + A_{22}P_{23}) & -(A_{21}P_{14} + A_{22}P_{24}) \\ 0 & & B_{33} & B_{34} \\ 0 & & B_{43} & B_{44} \end{pmatrix} \\ &= \begin{pmatrix} A_1 & -A_1P_1 \\ 0 & B_1 \end{pmatrix}. \end{aligned} \tag{9}$$

Since $A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B_1 = \begin{pmatrix} B_{33} & B_{34} \\ B_{43} & B_{44} \end{pmatrix}$ are invertible, $A - B$ is invertible. $\square$

Let $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$ and $\beta \neq 0$, $B \in \omega(Q)$ such that $B^2 = mB + nQ$ and $n \neq 0$. It should be noted, however, the invertibility of $P - Q$ is sufficient but not necessary. This can be demonstrated in the following example.

Example 1. Define the idempotents P, Q and operators A, B by

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $A \in \omega(P)$, $B \in \omega(Q)$, $A^2 = 3A - 2P$, $B^2 = 5B - 6Q$ and

$$A - B = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -2 \\ 0 & 2 & -1 \end{pmatrix} \quad \text{isinvertible.}$$

But

$$P - Q = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{is not invertible.}$$

Let P and Q be two idempotents. In [5], we have proved that if $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, then $P + Q$ is invertible if and only if $P - Q$ is invertible. So the following consequence is immediate.

Corollary 3.3. *Let P and Q in $\mathcal{B}(\mathcal{H})$ be two idempotents with $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$ and $\beta \neq 0$, $B \in \omega(Q)$ such that $B^2 = mB + nQ$ and $n \neq 0$. If one of $P \pm Q$, $1 - PQ$ and $P + Q - PQ$ is invertible then $A - B$ is invertible.*

Let us now consider the problem of finding the inverse of $A \in \omega(P)$. Let P be an idempotent and

$$P^{\perp} = P(P + P^* - I)^{-1} = (P + P^* - I)^{-1}P^* \tag{10}$$

be the range projection of P. By Lemma 2.1, item (3), the orthogonal projection $P^{\perp}$ satisfying $P^{\perp}P = P$ and $PP^{\perp} = P^{\perp}$. Clearly, if P has the form (1), then $P^{\perp} = I_{\mathcal{R}(P)} \oplus 0$.

Recall that an operator A has the Moore–Penrose inverse A^+ if and only if $\mathcal{R}(A)$ is closed and $A^+ = A^*(AA^*)^+ = (A^*A)^+A^*$ (see [[14], Theorem 1.1.2]). For $T \in \mathcal{B}(X)$, T has the Drazin inverse A^D if and only if $0 \notin acc[\sigma(T)]$ and in that case T^D is unique (see [3,14,15]). As a consequence of Lemma 2.1 and Theorem 3.1, we know that the generalized quadratic operator A is always Moore–Penrose invertible and Drazin invertible. Furthermore, A^{-1}, A^{+} and A^{D} can be represented by A and P respectively, as our next theorem domonstrates.

Theorem 3.4. *Let $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$. If $\beta \neq 0$, then*
(1) $A^{-1} = \frac{1}{\beta}(A - \alpha I)$ *if* $P = I$.
(2) $A^{+} = \frac{1}{\beta}P^{+}(A - \alpha P)P^{\perp}$.
(3) $A^{D} = \frac{1}{\beta}(A - \alpha I)P$.

Proof. (1) See the proof of Lemma 2.2, item (1).
(2) By the proof of Lemma 2.2 and (10), we have

$$\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta}(A_1 - \alpha I) & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{\beta}(A - \alpha P)P^{\perp}. \tag{11}$$

Hence

$$\begin{aligned} A^{+} &= A^*(AA^*)^{+} = \begin{pmatrix} A_1^* & 0 \\ P_1^*A_1^* & 0 \end{pmatrix} \begin{pmatrix} (A_1A_1^* + A_1P_1P_1^*A_1^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_1^* & 0 \\ P_1^*A_1^* & 0 \end{pmatrix} \begin{pmatrix} (A_1^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (I + P_1P_1^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{\beta}P^*(PP^*)^{+}(A - \alpha P)P^{\perp} \\ &= \frac{1}{\beta}P^{+}(A - \alpha P)P^{\perp}. \end{aligned}$$

(3) Similarly, we have

$$\begin{aligned} A^{D} &= \begin{pmatrix} A_1 & A_1P_1 \\ 0 & 0 \end{pmatrix}^{D} = \begin{pmatrix} A_1^{-1} & A_1^{-2}(A_1P_1) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} P \\ &= \frac{1}{\beta}(A - \alpha I)P^{\perp}P \\ &= \frac{1}{\beta}(A - \alpha I)P. \quad \square \end{aligned}$$

Note that the Drazin inverse of idempotent is itself. If $\alpha \neq 0$ and $\beta = 0$, then $\left(\frac{A}{\alpha}\right)^2 = \frac{A}{\alpha}$. Hence $A^{D} = \frac{A}{\alpha^2}$. Theorem 3.1, item (2) shows that, if $A \in \omega(P)$ is invertible, $A^{-1} = \frac{1}{\beta}(A - \alpha I)$ and $A^{-2} = -\frac{\alpha}{\beta}A^{-1} + \frac{1}{\beta}I$. Then $A^{-1} \in \omega(I)$. Furthermore, it is easy to verify that $A^{k+1} = \alpha_k A + \beta_k P$ and $A^{2k} = \lambda_k A^k + \mu_k P$ for some $\alpha_k, \beta_k, \lambda_k$ and $\mu_k \in \mathbb{C}$, depending on positive integer k. Hence each power of $A \in \omega(P)$ again belongs to this class (see [[8], Throrem 2] for the matrix case).

Using the notations from the proof of Theorems 3.2, we can check that, if $P - Q$ is invertible,

$$(P - Q)^{-1} = \begin{pmatrix} I & P_{13}M^*(I - P_{23}M^*)^{-1} & -P_{13}(M^*P_{23} - I)^{-1} & -P_{14} - P_{13}M^*(I - P_{23}M^*)^{-1}P_{24} \\ 0 & (I - P_{23}M^*)^{-1} & [M - (I - Q_{11}^{-1})P_{23}](M^*P_{23} - I)^{-1} & -(I - P_{23}M^*)^{-1}P_{24} \\ 0 & -M^*(I - P_{23}M^*)^{-1} & (M^*P_{23} - I)^{-1} & -M^*(I - P_{23}M^*)^{-1}P_{24} \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{12}$$

Let us now consider the problem of finding the inverse $A-B$ when $P-Q$ is invertible.

Theorem 3.5. *Let P and Q in $\mathcal{B}(\mathcal{H})$ be two idempotents, $A \in \omega(P)$ such that $A^2 = \alpha A + \beta P$ and $\beta \neq 0$, $B \in \omega(Q)$ such that $B_1^2 = mB + nQ$ and $n \neq 0$. If $P - Q$ is invertible, then*

$$(A-B)^{-1} = \frac{1}{\beta}(P-Q)^{-1}(A-\alpha P)P^{\perp}(I-P^{\perp}QP^{\perp})^{-1}(I-P^{\perp}Q) + \frac{1}{n}\left[(P-Q)^{-1}(P^{\perp}-P)+P^{\perp}-I\right]T(B-mQ)T(I-P^{\perp}),$$

where $T = [(I-P^{\perp})Q(I-P^{\perp})+P^{\perp}]^{-1}$.

Proof. Let G, S and $P^{\perp}$ be defined as (8) and (10). If $P-Q$ is invertible, by (3) and (12), we have

$$G = (I-P^{\perp}QP^{\perp})^{-1}(I-P^{\perp}Q) \quad \text{and} \quad S = (P-Q)^{-1}P^{\perp}+P^{\perp}-I.$$

By (11) and (6), we have $A_1^{-1} \oplus 0 = \frac{1}{\beta}(A-\alpha P)P^{\perp}$ and

$$\begin{pmatrix} 0 & -P_1B_1^{-1} \\ 0 & B_1^{-1} \end{pmatrix} = \frac{1}{n}(I-P)T(B-mQ)T(I-P^{\perp}),$$

where $T = [(I-P^{\perp})Q(I-P^{\perp})+P^{\perp}]^{-1}$. Note that $P^{\perp}(A-\alpha P) = P^{\perp}P(A-\alpha P) = A-\alpha P$ and $(I-P^{\perp})G = (I-P^{\perp})$. By (9),

$$\begin{aligned}
(A-B)^{-1} &= S\begin{pmatrix} A_1^{-1} & -P_1B_1^{-1} \\ 0 & B_1^{-1} \end{pmatrix}G \\
&= S\left[\frac{1}{\beta}(A-\alpha P)P^{\perp} + \frac{1}{n}(I-P)T(B-mQ)T(I-P^{\perp})\right]G \\
&= \left[(P-Q)^{-1}P^{\perp}+P^{\perp}-I\right] \times \frac{1}{\beta}(A-\alpha P)P^{\perp} \times \left[(I-P^{\perp}QP^{\perp})^{-1}(I-P^{\perp}Q)\right] \\
&\quad + \left[(P-Q)^{-1}P^{\perp}+P^{\perp}-I\right] \times \frac{1}{n}(I-P)T(B-mQ)T(I-P^{\perp}) \\
&= \frac{1}{\beta}(P-Q)^{-1}(A-\alpha P)P^{\perp}(I-P^{\perp}QP^{\perp})^{-1}(I-P^{\perp}Q) \\
&\quad + \frac{1}{n}\left[(P-Q)^{-1}(P^{\perp}-P)+P^{\perp}-I\right]T(B-mQ)T(I-P^{\perp}). \quad \square
\end{aligned}$$

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References

[1] M. Aleksiejczyk, A. Smoktunowicz, On properties of quadratic matrices, Math. Pannon. 11 (2000) 239–248.
[2] J. Benítez, N. Thome, Idempotency of linear combinations of an idempotent matrix and a t-potent matrix that commute, Linear Algebra Appl. 403 (2005) 414–418.
[3] C.Y. Deng, The Drazin inverses of sum and difference of idempotents, Linear Algebra Appl. 430 (2009) 1282–1291.
[4] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–416.
[5] H.K. Du, X.Y. Yao, C.Y. Deng, Invertibility of linear combinations of two idempotents, Proc. Amer. Math. Soc. 134 (2005) 1451–1457.
[6] Man-Duen Choi, Pei Yuan Wu, Convex combinations of projections, Linear Algebra Appl. 136 (1990) 25–42.
[7] Li Fang, Guoxing Ji, Yongfeng Pang, Maps preserving the idempotency of products of operators, Linear Algebra Appl. 426 (2007) 40–52.

[8] R.W. Farebrother, G. Trenkler, On generalized quadratic matrices, Linear Algebra Appl. 410 (2005) 244–253.
[9] Halim Özdemir, Ahmet YasarÖzban, On idempotency of linear combinations of idempotent matrices, Appl. Math. Comput. 159 (2004) 439–448.
[10] J.J. Koliha, V. Rakocevic, Range projections and the Moore–Penrose inverse in rings with involution, Linear and Multilinear Algebra 55 (2007) 103–112.
[11] H. Radjavi, P. Rosenthal, On commutators of idempotents, Linear and Multilinear Algebra 50 (2002) 121–124.
[12] Jinhua Zhou, Guorong Wang, Block idempotent matrices and generalized Schur complement, Appl. Math. Comput. 188 (2007) 246–256.
[13] Tzon-Tzer Lu, Sheng-Hua Shiou, Inverses of 2×2 block matrices, Comput. Math. Appl. 43 (2002) 119–129.
[14] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computions, Science Press, Beijing/New York, 2004.
[15] Y. Wei, Expressions for the Drazin inverse of a 2×2 block matrix, Linear and Multilinear Algebra 45 (1998) 131–146.